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2007 J. Phys. A: Math. Theor. 40 7233

(http://iopscience.iop.org/1751-8121/40/26/009)

IOP Publishing Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **40** (2007) 7233–7237 doi:10.1088/1751-8113/40/26/009

Isospectral drums in $\mathbb{R}^2$, involution graphs and Euclidean TI-domains

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Received 12 December 2006, in final form 16 May 2007
Published 12 June 2007
Online at stacks.iop.org/JPhysA/40/7233

Abstract
The widely investigated question 'Can one hear the shape of a drum?' which Kac posed in his published lecture (Kac 1966 *Am. Math. Mon.* **73** 1–23) was eventually answered negatively in Gordon *et al* (1992 *Invent. Math.* **110** 1–22) by construction of isospectral pairs in $\mathbb{R}^2$. Up to present, all known planar counter examples are constructed by a certain tiling method, and in this communication, we call such examples isospectral *Euclidean TI-domains*. From counter examples of this type, one can construct a pair of (finite) involution graphs. In this communication, we address the question as to how the isospectrality of the domains for the Laplacian influences the cospectrality of the involution graphs.

PACS number: 02.30.Zz

1. Introduction and recognition

A celebrated inverse problem posed by Kac in his published lecture [8] asks whether simply connected domains in $\mathbb{R}^2$ for which the sets $\{\lambda_n \parallel n \in \mathbb{N}\}$ of solutions (eigenvalues) of the stationary Schrödinger equation

$$(\Delta + \lambda)\Psi = 0 \qquad \text{with} \qquad \Psi|_{\text{boundary}} = 0$$

coincide, are necessarily congruent.

Counter examples were constructed to the analogous question on Riemannian manifolds (cf Brooks [2]), but for Euclidean domains the question appeared to be much harder. Gordon, Webb and Wolpert constructed a pair of simply connected non-isometric Euclidean isospectral domains—also called 'planar isospectral pairs' or 'isospectral billiards'—in [7]. All the known (counter) examples up to now can be found in the paper by Buser, Conway, Doyle and Semmler [3]. (A survey on the broad theory developed after Kac's paper is [6].)

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Up to present, all known planar counter examples are constructed by a certain tiling method, and we call such examples isospectral *Euclidean TI-domains*. Even more, all these examples have the property that they are *transplantable*. From counter examples of this type, one can construct a pair of (finite) involution graphs. In this communication, we address the question as to how the isospectrality of the domains for the Laplacian influences the cospectrality of the involution graphs.

Essentially, up to homothecy, only a finite number of examples were constructed if one fixes the congruence class of the base tile [3].

So it is a crucial question as to whether there are efficient ways to recognize isospectral billiards. Here, we look at this question for billiards which arise by the aforementioned tiling method—so for Euclidean TI-domains. Also, we only consider TI-domains for which the base tile is a triangle, because these are generally considered as being the prototypical base tiles.

We note that in the papers [5, 11–13] such a recognition theory was developed from the group-theoretical point of view: if one considers two isospectral Euclidean TI-domains which are *transplantable*, they are associated with the same operator group (which is the group generated by the involutions defined by the associated involution graph). In all known examples, the operator groups are of a very restricted type: they are all isomorphic to the classical group $\mathbf{PSL}_n(q)$, where $(n, q) \in \{(3, 2), (3, 3), (4, 2), (3, 4)\}$. In [13], it was eventually shown that if (D_1, D_2) is a pair of non-congruent planar isospectral domains constructed from unfolding any polygonal base tile and with associated operator group $\mathbf{PSL}_n(q)$, $n \geqslant 2$, then (n, q) belongs to $\{(3, 2), (3, 3), (4, 2), (3, 4)\}$.

2. Involution graphs and TI-domains

2.1. Tiling

All known isospectral billiards can be obtained by unfolding polygonal-shaped tiles, but essentially one can only consider triangles. The way the tiles are unfolded can be specified by three permutation $N \times N$-matrices $M^{(\mu)}$, $1 \leqslant \mu \leqslant 3$ and $N \in \mathbb{N}$, associated with the three sides of the triangle:

- $M^{(\mu)}_{ij} = 1$ if tiles i and j are glued by their side μ;
- $M^{(\mu)}_{ii} = 1$ if the side μ of tile i is the boundary of the billiard, and
- 0 otherwise.

The number of tiles is, of course, N.

One can sum up the action of the $M^{(\mu)}$ in a graph with coloured edges: each copy of the base tile is associated with a vertex, and vertices i and j, $i \neq j$, are joined by an edge of colour μ if and only if $M^{(\mu)}_{ij} = 1$. The consideration of the coloured graph as such defined was first made by Okada and Shudo in [10]. In the same way, in the second member of the pair, the tiles are unfolded according to permutation matrices $N^{(\mu)}$, $1 \leqslant \mu \leqslant 3$. We call such a coloured graph an *involution graph* for reasons to be explained later in this section. If D is a Euclidean TI-domain with base tile a triangle, and $\mathbf{M} = \{M^{(\mu)} \| \mu \in \{1, 2, 3\}\}$ is the set of associated permutation matrices (or, equivalently, the associated colouring), denote by $\Gamma(D, \mathbf{M})$ the corresponding involution graph.

The following proposition is easy but fundamental.

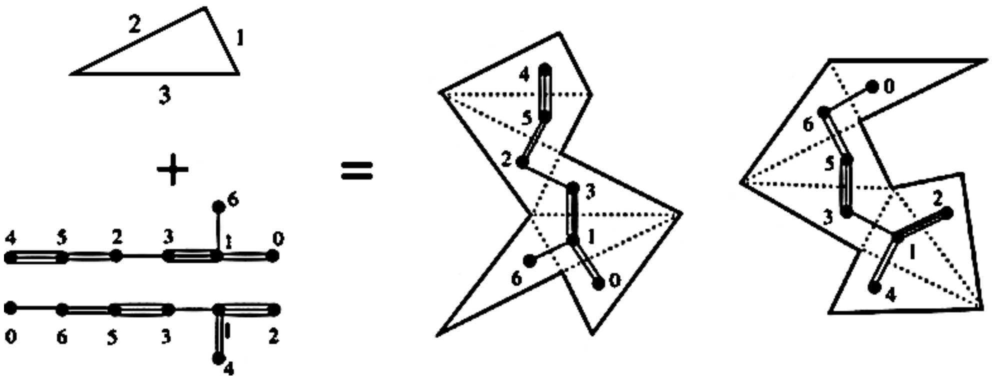

Figure 1. The involution graphs corresponding to a pair of isospectral billiards: if we label the sides of the triangle by $\mu = 1, 2, 3$, the unfolding rule by symmetry with respect to side μ can be represented by edges made of μ braids in the graph. (From a given pair of graphs, one can construct infinitely many pairs of isospectral billiards by applying the unfolding rules to any triangle.)

Proposition 2.1. *Let D be a Euclidean TI-domain with base tile a triangle, and let $\mathbf{M} = \{M^{(\mu)} \parallel \mu \in \{1, 2, 3\}\}$ be the set of associated permutation matrices. Then*

$$\sum_{\mu=1}^{3} M^{(\mu)} - \Delta\left(\sum_{\mu=1}^{3} M^{(\mu)}\right)$$

is the adjacency matrix of $\Gamma(D, \mathbf{M})$, where $\Delta^M = \Delta\left(\sum_{\mu=1}^{3} M^{(\mu)}\right)$ is defined by $\Delta^M_{ii} = \left(\sum_{\mu=1}^{3} M^{(\mu)}\right)_{ii}$ for all i, and $\Delta^M_{ij} = 0$ if $i \neq j$. □

2.2. Transplantability

Two billiards are said to be *transplantable* if there exists an invertible matrix T—the *transplantation matrix*—such that

$$TM^{(\mu)} = N^{(\mu)}T \qquad \text{for all} \quad \mu.$$

If the matrix T is a permutation matrix, the two domains would just have the same shape. One can show that transplantability implies isospectrality.

If Ψ_1 is an eigenfunction of the first billiard and $\Psi_1^{(i)}$ is its restriction to triangle i, then one can build an eigenfunction Ψ_2 of the second billiard by taking

$$\Psi_2^{(i)} = \sum_j T_{ij}\Psi_1^{(j)}.$$

2.3. Involutions and operator groups

Suppose D is a Euclidean TI-domain on N base triangles, and let $M^{(\mu)}$, $\mu \in \{1, 2, 3\}$, be the corresponding permutation $N \times N$-matrices. Define involutions $\theta^{(\mu)}$ on a set X of N letters $\Delta_1, \Delta_2, \ldots, \Delta_N$ (indexed by the base triangles) as follows: $\theta^{(\mu)}(\Delta_i) = \Delta_j$ if $M_{ij}^{(\mu)} = 1$ and $i \neq j$. In the other cases, Δ_i is mapped onto itself. Then clearly, $\langle\theta^{(\mu)} \| \mu \in \{1, 2, 3\}\rangle$ is a transitive permutation group on X, which we call the *operator group* of D. More details on operator groups can be found in [13].

2.4. TI-domains and graph cospectrality

Two graphs are *cospectral* if their adjacency matrices have the same spectrum.

The question we want to pose is:

Question. *Let (D_1, D_2) be a pair of nonisometric isospectral Euclidean TI-domains, and let $\Gamma(D_1) = \Gamma(D_1, \{M^{(\mu)} \parallel \mu \in \{1,2,3\}\})$ and $\Gamma(D_2) = \Gamma(D_2, \{N^{(\mu)} \parallel \mu \in \{1,2,3\}\})$ be the corresponding involution graphs. Are $\Gamma(D_1)$ and $\Gamma(D_2)$ cospectral?*

Note that we do not ask the domains to be transplantable.

We now show that, maybe surprisingly, the answer is 'yes' when the domains are transplantable.

Proof. Define, for $\mu = 1, 2, 3$, $M_*^{(\mu)}$ as the matrix which has the same entries as $M^{(\mu)}$, except on the diagonal, where it only has zeros. Define matrices $N_*^{(\mu)}$ analogously. Suppose that $TM^{(\mu)}T^{-1} = N^{(\mu)}$ for all μ.

Note the following properties:

- $M_*^{(\mu)}$ and $N_*^{(\mu)}$, $\mu = 1, 2, 3$, are symmetric (0, 1)-matrices, with at most one 1 entry on each row;
- $[M_*^{(\mu)}]^m = M_*^{(\mu)}$ if the natural number m is odd and $[M_*^{(\mu)}]^m = \mathbb{I}_M^{(\mu)}$, where $\left[\mathbb{I}_M^{(\mu)}\right]_{ii} = 1$ if there is a 1 on the ith row of $M_*^{(\mu)}$, and 0 otherwise, if m is even, $\mu = 1, 2, 3$, and similar properties hold for the $N_*^{(\mu)}$;
- $\mathrm{Tr}(M_*^{(i)}M_*^{(j)}) = \mathrm{Tr}(M_*^{(j)}M_*^{(i)}) = 0$ for $i \neq j$ and $\mathrm{Tr}(N_*^{(i)}N_*^{(j)}) = \mathrm{Tr}(N_*^{(j)}N_*^{(i)}) = 0$ for $i \neq j$;
- $\mathrm{Tr}(M_*^{(i)}M_*^{(j)}M_*^{(k)})$ and $\mathrm{Tr}(N_*^{(i)}N_*^{(j)}N_*^{(k)})$ are independent of the permutation (ijk) of (123) (this is because the individual matrices are symmetric);
- the value of all traces in the previous property is 0 (note that if $\{i, j, k\} = \{1, 2, 3\}$, such a trace equals 0 since the existence of a nonzero diagonal entry of $M_*^{(i)}M_*^{(j)}M_*^{(k)}$, respectively $N_*^{(i)}N_*^{(j)}N_*^{(k)}$, implies $\Gamma(D_1)$, respectively $\Gamma(D_2)$, to have closed circuits of length 3);
- $\sum_{\mu=1}^{3} M_*^{(\mu)} = \sum_{\mu=1}^{3} M^{(\mu)} - \Delta\left(\sum_{\mu=1}^{3} M^{(\mu)}\right)$ and $\sum_{\mu=1}^{3} N_*^{(\mu)} = \sum_{\mu=1}^{3} N^{(\mu)} - \Delta\left(\sum_{\mu=1}^{3} N^{(\mu)}\right)$.

Put $A = \sum_{\mu=1}^{3} M_*^{(\mu)}$, the adjacency matrix of $\Gamma(D_1)$, and $B = \sum_{\mu=1}^{3} N_*^{(\mu)}$, the adjacency matrix of $\Gamma(D_2)$.

Consider a natural number $n \in \mathbb{N}_0$. Then keeping the previous properties in mind, it follows that

$$\mathrm{Tr}(A^n) = \mathrm{Tr}(B^n)$$

whence by [4, lemma 1] the adjacency matrices of $\Gamma(D_1)$ and $\Gamma(D_2)$ have the same spectrum. □

Note that from our result, we have, by [9], the following implication for 'starlike trees' [9]:

Corollary 2.2. *If $\Gamma(D_1)$ and $\Gamma(D_2)$ are starlike and D_1 and D_2 are transplantable, they are isomorphic.* □

Acknowledgments

This communication was written while the author was visiting the Discrete Mathematics group of the Technical University of Eindhoven, The Netherlands, whose hospitality he gratefully acknowledges.

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